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Introduction to Open Multi-Agent Systems - Part II

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Outline

- ① Challenges in Open Multi-Agent Systems
- ② Open Average Consensus: how to preserve invariant quantities in OMAS
- ③ Convergence and Stability in Contractive Open Multi-Agent Systems

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Graph-theoretical notation for Open MAS

At each time k , an Open Multi-Agent System is described by a time-varying graph

$$\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k),$$

where the set of agents is time-varying. $\mathcal{V}_k$, $n_k = |\mathcal{V}_k|$.

Edge set

$$\mathcal{E}_k \subseteq \mathcal{V}_k \times \mathcal{V}_k.$$

Neighbor set $\mathcal{N}_i(k) = \{j \in \mathcal{V}_k : \{i, j\} \in \mathcal{E}_k\}$.

Only agents in $\mathcal{V}_k$ participate in the dynamics and exchange information.

Key difference from classical MAS: Changing the number of agents changes the cardinality of the state space in the network system $x(k) \in \mathbb{R}^{pn_k}$.



- $R_k = V_k \cap V_{k+1}$, i.e., remaining nodes that belong to both V_k and V_{k+1} ;
- $D_k = V_k \setminus V_{k+1}$, i.e., departing nodes that belong to V_k but not to V_{k+1} ;
- $A_k = V_{k+1} \setminus V_k$, i.e., arriving nodes that belong to V_{k+1} but not to V_k .

$$V_{k+1} = R_k \cup A_k, \quad V_k = R_k \cup D_k.$$

Analyzing the dynamics of an Open MAS

Arrivals

If a new agent $i \in \mathcal{A}_{k+1}$ joins, we must define its state at the time of joining:

$$x_i(k+1) = ?$$

Typical choices:

- arbitrary initialization;
- initialization from local measurements;
- initialization from neighbors;
- stored memory from a previous participation.

Departures

If an agent $i \in \mathcal{D}_k$ leaves, we must consider what happens to its state:

$$x_i(k) \longrightarrow ?$$

Key questions:

- Is its information discarded?
- Is some quantity preserved?
- Do neighbors compensate its departure?
- Does the objective change?

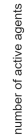
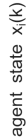
Key issue: an OMAS requires both a dynamical update rule and an arrival/departure mechanism.

[illegible]

Let us consider what happens to the classical Laplacian based discrete-time consensus protocol in a realistic setting where agents join and leave the network.

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in N_i} l_{ij}(x_j(k) - x_i(k)), \quad i, j \in \mathcal{V}_k, k \in \mathbb{N}.$$

- The graph is a random graph with 20 agents initially present.
- When an agent departs its state is simply removed from the network.
- When an agent arrives, it enters with a state chosen at random between $[0, 10]$ and it randomly connects to a set of agents.
- One agent arrives with probability $p = 0.6$ at each time step.
- One agent departs with probability $p = 0.3$ at each time step.



First challenge in Open MAS: preserving invariants

Classical average consensus

For a time-varying undirected network: $x(k+1) = (I - \varepsilon L(k))x(k)$

Total mass (sum of states) is preserved: $\mathbf{1}^\top x(k+1) = \mathbf{1}^\top x(k)$.

Therefore the average is invariant: $\bar{x}(k) = \frac{1}{n} \sum_{i=1}^n x_i(k) = \bar{x}(0)$.

Similar issue in many Distributed Consensus-based Optimization and Control methods

What changes in an Open MAS?

When agents join or leave,

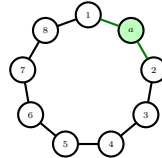
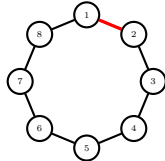
$$\mathcal{V}_k \neq \mathcal{V}_{k+1}, \quad x(k) \in \mathbb{R}^{|\mathcal{V}_k|}, \quad x(k+1) \in \mathbb{R}^{|\mathcal{V}_{k+1}|}.$$

The quantity $\sum_{i \in \mathcal{V}_k} x_i(k)$ is generally not preserved unless arrivals and departures are explicitly handled.

Key challenge: design join/leave mechanisms that keep the desired invariant quantities invariant.

Let us assume that new agents enter the network with a state value already present in the network at time k .

Let us just ask that the network achieves consensus on any value, would our Laplacian based consensus protocol converge?



- $$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in N_i} l_{ij}(x_j(k) - x_i(k)), \quad i, j \in \mathcal{R}_k,$$

$$x_a(k+1) = 0, \quad a \in A_k \quad (1)$$

$$x_{\mathcal{V}_{k+1}}(k+1) = (I_n - \varepsilon L(k))x_{\mathcal{V}_k}(k),$$

The arriving agent initializes as zero:

$$x_a(k+1) = 0.$$

Therefore, the state transition matrix is rectangular:

$$x_{\mathcal{V}_{k+1}}(k+1) = W_k x_{\mathcal{V}_k}(k),$$

$$W_k = \begin{bmatrix} I_n - \varepsilon L(k) \\ \mathbf{0}^T \end{bmatrix} \in \mathbb{R}^{(n+1) \times n}.$$

Singular values instead of eigenvalues

Since $W(k) \in \mathbb{R}^{(n+1) \times n}$ is rectangular, its eigenvalues are not defined.

Definition

The singular values of $A \in \mathbb{R}^{m \times n}$ are

$$\sigma_i(A) = \sqrt{\lambda_i(A^\top A)}, \quad i = 1, \dots, \min\{m, n\}.$$

Equivalently, they are the diagonal entries of the singular value decomposition:

$$A = U\Sigma V^{\top}.$$

Relation with eigenvalues

If $A = A^\top$, then $\sigma_i(A) = |\lambda_i(A)|$.

For symmetric square matrices, singular values measure the magnitude of eigenvalues.

Relation with norms

The induced 2-norm is

$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_{\max}(A)$. Therefore, singular values quantify the maximum stretching of vectors by A .

If
$$W_k = \begin{bmatrix} I - \varepsilon L \\ 0 \dots 0 \end{bmatrix} \in \mathbb{R}^{(n+1) \times n}.$$

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then, since $L = L^\top$,

$$W^\top(k)W(k) = (I - \varepsilon L(k))^\top (I - \varepsilon L(k)) = (I - \varepsilon L)^2.$$

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If ($d_{max} = 2$ for a ring graph)

$$0 < \varepsilon \leq \frac{1}{2d_{max}} = \frac{1}{4},$$

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then the eigenvalues of $I - \varepsilon L$ are positive. Therefore,

$$\sigma_i(W(k)) = 1 - \varepsilon \lambda_i(L), \quad i = 1, \dots, n.$$
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Since $\lambda_1(L) = 0$, $\sigma_1(W(k)) = 1$, and $\sigma_2(W(k)) = 1 - \varepsilon\lambda_2(L)$. Also, $W(k) \begin{bmatrix} \mathbf{1}^\top & 0 \end{bmatrix}^\top = \begin{bmatrix} \mathbf{1}^\top & 0 \end{bmatrix}^\top$ and $W^T(k) \begin{bmatrix} \mathbf{1}^\top & 0 \end{bmatrix}^\top = \begin{bmatrix} \mathbf{1}^\top & 0 \end{bmatrix}^\top$

Also, $W(k) \begin{bmatrix} \mathbf{1}^\top \mathbf{0} \end{bmatrix}^\top = \begin{bmatrix} \mathbf{1}^\top \mathbf{0} \end{bmatrix}^\top$ and $W^T(k) \begin{bmatrix} \mathbf{1}^\top \mathbf{0} \end{bmatrix}^\top = \begin{bmatrix} \mathbf{1}^\top \mathbf{0} \end{bmatrix}^\top$

Natural evolution and loss of uniform contraction

$$\prod_{\tau=0}^{k-1} \sigma_2(W_\tau) = \prod_{\tau=0}^{k-1} \left[1 - 4\epsilon \sin^2 \left(\frac{\pi}{n_0 + \tau} \right) \right].$$

Since

$$4\varepsilon \sin^2\left(\frac{\pi}{n_0 + \tau}\right) \sim \frac{4\varepsilon \pi^2}{(n_0 + \tau)^2}, \quad \text{and} \quad \sum_{\tau=0}^{\infty} \frac{1}{(n_0 + \tau)^2} < \infty,$$

the infinite product converges to a positive constant:

$$\prod_{\tau=0}^{\infty} \sigma_2(W_\tau) = \beta > 0.$$

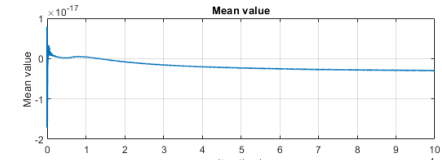
Basic intuition

The accumulated contraction over infinite iterations is finite even if the dynamics contracts at each step:

$$\prod_{\tau=0}^{\infty} \sigma_2(W_\tau) \not\rightarrow 0.$$

This despite a constant ε and no vanishing elements of W .

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Open Average Consensus

Consider an Open MAS described by the undirected time-varying graph

$$\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k).$$

Between k and $k + 1$, define

$$\mathcal{R}_k = \mathcal{V}_k \cap \mathcal{V}_{k+1}, \quad \mathcal{A}_k = \mathcal{V}_{k+1} \setminus \mathcal{V}_k, \quad \mathcal{D}_k = \mathcal{V}_k \setminus \mathcal{V}_{k+1}.$$

 $\mathcal{R}_k$: remaining agents, $\mathcal{A}_k$: arriving agents, $\mathcal{D}_k$: departing agents.

Open Average Consensus

For each agent i , define the indicators

$$\phi_i(k) = \begin{cases} 1, & i \in \mathcal{R}_k, \\ 0, & \text{otherwise,} \end{cases} \quad \psi_i(k) = \begin{cases} 1, & i \in \mathcal{A}_k, \\ 0, & \text{otherwise.} \end{cases}$$

Local interaction protocol:

$$x_i(k+1) = \psi_i(k)\bar{x}_i + \phi_i(k)x_i(k) + \underbrace{\phi_i(k) \varepsilon \left(\sum_{j \in N_k \cap \mathcal{R}(k)} l_{ij}(k)(x_j(k) - x_i(k)) + \sum_{j \in N_k \cap \mathcal{D}(k)} z_{ij}(k) \right)}_{u_i(k)}. \quad (2)$$

Each agent maintains and updates auxiliary variables $z_{ij}(k)$:

$$z_{ij}(t+1) = \phi_i(k)\phi_j(k)(z_{ij}(k) - \varepsilon l_{ij}(k)(x_j(k) - x_i(k))). \quad (3)$$

$z_{ij}(k+1) = 0$ whenever either $i \notin \mathcal{R}(k)$ or $j \notin \mathcal{R}(k)$ (or both).

Protocol interpretation: remaining agents

If agent i remains in the network, then

$$i \in \mathcal{R}_k.$$

If all its neighbors also remain, the update reduces to the standard consensus step:

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in \mathcal{N}_i(k) \cap \mathcal{R}_k} l_{ij}(k)(x_j(k) - x_i(k)).$$

The auxiliary variables accumulate exchanged mass:

$$z_{ij}(k+1) = z_{ij}(k) - \varepsilon l_{ij}(k)(x_j(k) - x_i(k)).$$

Interpretation: z_{ij} records the cumulative effect of past interactions across edge $\{i, j\}$.

Protocol interpretation: arrivals

If agent i joins the network, then

$$i \in \mathcal{A}_k.$$

It initializes its state as

$$x_i(k+1) = \bar{x}_i.$$

Auxiliary variables in each are defined and initialized as

$$z_{ij}(k+1) = 0, \quad j \in \mathcal{N}_i(k+1).$$

Interpretation: an arriving agent injects its own value into the active network, e.g., outcome of a measurement or physical process.

Protocol interpretation: departures

If agent $i \in \mathcal{R}_k$ remains but a neighbor $j \in \mathcal{D}_k$ departs, then

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in \mathcal{N}_i(k) \cap \mathcal{R}_k} l_{ij}(k)(x_j(k) - x_i(k)) \\ + \sum_{j \in \mathcal{N}_i(k) \cap \mathcal{D}_k} z_{ij}(k).$$

For the edges that remain active,

$$z_{ij}(k+1) = z_{ij}(k) - \varepsilon l_{ij}(k)(x_j(k) - x_i(k)), \quad j \in \mathcal{N}_i(k) \setminus \mathcal{D}_k.$$

Key idea

When an agent departs, neighboring agents use $z_{ij}(k)$ to cancel the effect of previous interactions with the departing agent.

Sum of states preservation and convergence

Theorem (Average preservation)

Consider an OMAS executing the average-preserving consensus protocol over an undirected time-varying graph

$$\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k).$$

Then, for all $k \geq 0$,

$$\sum_{i \in \mathcal{V}_k} x_i(k) = \sum_{i \in \mathcal{V}_k} \bar{x}_i.$$

Zoreh Al Zahra Sanai Dashti, Gabriele Oliva, Carla Seatzu, Andrea Gasparri, and Mauro Franceschelli "Distributed Mode Computation in Open Multi-Agent Systems", IEEE Control Systems Letters, 2022.

At $k = 0$, it holds $V_0 = A_0$, therefore

$$\sum_{i \in V_0} x_i(0) = \sum_{i \in V_0} \bar{x}_i,$$

For $k \geq 0$, by substituting the protocol state updates for all $i \in R_k$:

$$\begin{aligned} x_i(k+1) + \sum_{j \in N_i(k+1)} z_{ij}(k+1) &= x_i(k) + \sum_{j \in N_i \cap R_k} \varepsilon l_{ij}(x_i(k) - x_j(k)) + \sum_{j \in N_i(k) \cap_k} z_{ij}(k) \\ &\quad + \sum_{j \in \mathcal{N}_i(k) \cap R_k} z_i(k) - \sum_{j \in \mathcal{N} \cap R_k} \varepsilon l_{ij}(x_i(k) - x_j(k)) \\ &= x_i(k) + \sum_{j \in \mathcal{N}_k} z_i(k). \end{aligned}$$

By defining k_i^* as the time instant at which agent i joins the network, it holds

$$x_i(k+1) + \sum_{j \in N_i(k)} z_{ij}(k+1) = x_i(k_i^*) + \sum_{j \in N_i(k_i^*)} z_{ij}(k_i^*).$$



By construction, all $z_{ij}(k_i^*) = 0$ and $x_i(k_i^*) = \bar{x}_i$; thus, for all $i \in R_k$:

$$x_i(k+1) + \sum_{j \in N_i(k)} z_{ij}(k+1) = \bar{x}_i.$$

For all $i \in A_k$, by definition $x_i(k+1) = \bar{x}_i$ and $z_{ij}(k+1) = 0$; therefore,

$$x_i(k+1) + \sum_{j \in N_i(k+1)} z_{ij}(k+1) = \bar{x}_i$$

Summing over all $i \in R_{kk} = V_{k+1}$, we obtain

$$\sum_{i \in V_{k+1}} x_i(k+1) + \sum_{i \in V_{k+1}} \sum_{j \in N_i(k+1)} z_{ij}(k+1) = \sum_{i \in V_{k+1}} \bar{x}_i.$$

Finally, by construction $z_{ij}(k) = -z_{ji}(k)$, and thus for all k

$$\sum_{i \in V_k} \sum_{j \in N_i(k)} z_{ij}(k) = 0.$$



What happens when agents stop arriving and departing?

If eventually agents stop arriving and departing, the state update evolves according to :

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in \mathcal{N}_i(k) \cap \mathcal{R}_k} l_{ij}(k)(x_j(k) - x_i(k)).$$

Which yields the familiar discrete-time consensus dynamics $x(k+1) = (I - \varepsilon L(k))x(k)$ which converges exponentially fast to the average of its sum of states.

Numerical example of Open Average Consensus: Setup

- Initial number of agents: 30
- Every 50 iterations one agent joins or leaves with probability $p = 0.5$
- The initial states of the agents are randomly selected in the interval $[0, 1]$
- Each agent that arrives has a random state selected in the interval $[0, 1]$

Open Average Consensus: sample of references and assumptions II

Reference	Graph topology	Join/leave model	Mechanism and main message
C. N. Hadjicostis and A. D. Domínguez-García, "Distributed Average Consensus in Open Multi-Agent Systems," IEEE Conference on Decision and Control, 2024.	Directed graph induced by the active agents; nominal digraph is strongly connected.	Finite universe of possible agents. Agents may deactivate and reactivate, possibly multiple times. Departures are detected through absence.	Running-sum ratio consensus. If the active set eventually stabilizes, remaining agents converge to the average of their quantities of interest.
E. Makridis, A. Grammenos, G. Oliva, E. Kalyvianaki, C. N. Hadjicostis, and T. Charalambous, "Average Consensus over Directed Networks in Open Multi-Agent Systems with Acknowledgement Feedback," IEEE Conference on Decision and Control, 2024.	Directed, possibly unbalanced graph; active network assumed strongly connected.	Finite set of possible agents. Agents can join or leave at will. Acknowledgement feedback is required.	OPENRC ratio-consensus algorithm. Handles directed unbalanced networks and preserves mass during quiescent epochs between topology changes.
A. Miele, M. Lippi, M. Franceschelli, C. N. Hadjicostis, and A. Gasparri, "Consensus in Open Multi-Agent Systems over Directed Graphs with Asynchronous Interactions and Unannounced Departures," manuscript, 2026.	Directed time-varying graph; uniformly strongly connected over bounded time windows.	Agents join and leave; departures are unannounced. Agents operate asynchronously with local clocks.	Extended ratio consensus with watchdog timers and interaction histories. Handles asynchronous communication and unannounced departures.

Gossip-Based Estimation of Centroid and Common Reference Frame in OMAS

Problem:

Centroid and common reference frame estimation in GPS-denied environments. The arrival and departure of agents causes the centroid to shift dynamically, introducing a bias that static algorithms cannot handle.

Method:

Asynchronous gossip protocol that relies only on local relative measurements. To handle network fluctuations, it introduces interaction timers to detect unannounced departures and auxiliary memory variables to compensate for them.

[autoplay]0.56videos/gossip.mp4

A. Miele, M. Franceschelli and A. Gasparri, "Gossip-Based Estimation of Centroid and Common Reference Frame in Open Multi-Agent Systems," in IEEE Control Systems Letters, vol. 9, pp. 823-828, 2025.

Loitering Control for Open Drone Swarms

Problem:

Coordinated loitering (sustained orbital motion) in drone swarms fails under standard static strategies when the network is open, as the swarm's centroid shifts continuously with agent arrivals and departures.

Method:

The framework integrates a distributed consensus protocol for real-time local centroid estimation with a vector-field control law (attractive and repulsive fields) to maintain stable circular orbits and avoid collisions.

[autoplay]0.56videos/loitering.mp4

A. Miele, M. Franceschelli and A. Gasparri, "Distributed Consensus-Based Loitering Control for Open Multi-Agent Drone Swarms," in IFAC World Congress, 2026.

Potential-Based Coordination in OMAS

Problem:

Traditional potential frameworks rely on the invariance of the sum of the states to guarantee stability. In OMAS, this property is violated because departing agents leave behind uncompensated residual energy from broken pairwise interactions. This loss of symmetry causes a persistent global drift destroying coordination.

[autoplay]0.56videos/potential.mp4

Method:

The approach uses a distributed protocol with cumulative dissipation states to manage residual energy from departing agents, coupled with a distributed min-consensus algorithm for finite-time step size computation.

A. Miele, M. Lippi, D. Deplano, M. Franceschelli and A. Gasparri, "Distributed Potential-Based Coordination within Open Multi-Agent Systems," in *IEEE Control Systems Letters*, 2026.

Outline

- 1 Challenges in Open Multi-Agent Systems
- 2 Open Average Consensus: how to preserve invariant quantities in OMAS
- 3 Convergence and Stability in Contractive Open Multi-Agent Systems

- $$x_k \in \mathbb{R}^{V_k}, \quad k \in \mathbb{Z}_{\geq 0}.$$

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- Each agent has scalar state $x_i \in \mathbb{R}$.
- The states of the departing agents in the set D_k are simply deleted from the set of states and are not present in $x_{k+1} \in \mathbb{R}^{V_{k+1}}$.
- The states of the arriving agents in A_k need to be “initialized” according to some rule.
- For all $i \in R_k$, a causal evolution law takes the form

$$x_{k+1}^i = \bar{f}^i(x_k, V_k, E_k, u_k).$$

- If $V_k = V_{k+1}$, then

$$x_{k+1} = \bar{f}(x_k, V_k, E_k, u_k).$$

Agents may join or leave the network freely, with no upper bound on $|V_k|$.

- $R_k = V_k \cap V_{k+1}$, i.e., remaining nodes that belong to both V_k and V_{k+1} ;
- $D_k = V_k \setminus V_{k+1}$, i.e., departing nodes that belong to V_k but not to V_{k+1} ;
- $A_k = V_{k+1} \setminus V_k$, i.e., arriving nodes that belong to V_{k+1} but not to V_k .

Notably, by definition, we have that

$$V_{k+1} = R_k \cup A_k, \quad V_k = R_k \cup D_k.$$

An Intuitive example: Open Proportional Dynamic Consensus

Scenario: In a network of agents each receives an exogenous signal u_k^i

Algorithm objective: Each agent needs to track the time-varying value of the average of network inputs:

$$\frac{1}{|V_k|} \sum_{i \in V_k} u_k^i$$

with bounded error in an OMAS.

$$x_{k+1}^i = x_k^i - \alpha(x_k^i - u_k^i) - \varepsilon \sum_{j \in N_k^i} (x_k^i - x_k^j), \quad i \in R_k, \quad (4a)$$

$$x_{k+1}^i = u_{k+1}^i, \quad i \in A_k. \quad (4b)$$

$$\varepsilon > 0, \quad \alpha \in (0, 0.5).$$

M. Franceschelli and P. Frasca, "Proportional Dynamic Consensus in Open Multi-Agent Systems," IEEE Conference on Decision and Control, 2018.

Reduction to Proportional Dynamic Consensus

If $V_{k+1} = V_k$, then OPDC reduces to Proportional Dynamic Consensus. Namely, it reduces to the update in (4a), written in vector form as

$$\begin{aligned} x_{k+1} &= x_k - \alpha(x_k - u_k) - \varepsilon L_k x_k \\ &= ((1 - \alpha)I - \varepsilon L_k)x_k + \alpha u_k \\ &= P_k x_k + \alpha u_k, \end{aligned} \tag{5}$$

where

$$P_k = (1 - \alpha)I - \varepsilon L_k.$$

Interpretation

- The term $-\varepsilon L_k x_k$ enforces agreement, while $-\alpha(x_k - u_k)$ pulls each state toward its local input
- $\alpha(x_k - u_k)$ and can be interpreted as the gradient of $\frac{\alpha}{2} \|x_k - u_k\|_2^2$.
- If we were able to enforce a consensus constraints $x_i = x_j = c$ for all $i, j \in V_k$ the optimal solution c of $\min_c \|c - u_k\|_2^2$ is the average of u_k .

Classical Dynamic Average Consensus

M. Zhu and S. Martínez, "Discrete-time dynamic average consensus," *Automatica*, 2010.

M. Franceschelli and A. Gasparri, "Multi-stage discrete time and randomized dynamic average consensus," *Automatica*, 2019.

S. S. Kia, B. Van Scoy, J. Cortés, R. A. Freeman, K. M. Lynch, and S. Martínez, "Tutorial on Dynamic Average Consensus: The Problem, Its Applications, and the Algorithms," IEEE Control Systems Magazine, 2019.

Proportional Dynamic Average Consensus: Property 1

Property 1: steady state of PDC

If $\mathcal{G}$ is undirected and connected and

$$\alpha \in (0, 1 - \varepsilon d_{\max}), \quad \varepsilon \in \left(0, \frac{1}{2d_{\max}}\right).$$

If $u(k) = u$ is held constant, then the steady-state is

$$x^e = (\alpha I + \varepsilon L)^{-1} \alpha u,$$

Proportional Dynamic Average Consensus: Property 2

Relationship between x^e and the average of the input signals, $\bar{u}$ and disagreement $\hat{u} = (\mathbf{u} - \bar{u} \mathbf{1})$.

$$x^e = (I - P)^{-1} \alpha u = (\alpha I + \varepsilon L)^{-1} \alpha (\bar{u} \mathbf{1} + \hat{u}).$$

Since $(\alpha I + \varepsilon L)^{-1} \bar{u} \mathbf{1} = \frac{1}{\alpha} \bar{u} \mathbf{1}$ for any L_k we can write

$$x^e - \bar{u} \mathbf{1} = \alpha (\alpha I + \varepsilon L)^{-1} \hat{u}.$$

Since the eigenvector corresponding to the largest eigenvalue of $(\alpha I + \varepsilon L)^{-1}$ is $\mathbf{1}$ and $\mathbf{1}^T \hat{u} = 0$, it holds

$$\|x^e - \bar{u} \mathbf{1}\|_2 = \|\alpha (\alpha I + \varepsilon L)^{-1} \hat{u}\|_2 \leq \frac{1}{1 + \frac{\varepsilon \lambda_2(L)}{\alpha}} \|\hat{u}\|_2, \quad (6)$$

Parameters α and ε tune the error and convergence rate.

A cascade of m PDC protocols can reduce the error to $\left(\frac{1}{1 + \frac{\varepsilon \lambda_2(L)}{\alpha}} \right)^m$

Franceschelli and A. Gasparri, "Multi-stage discrete time and randomized dynamic average consensus," *Automatica*, 2019.

We need three ingredients to analyze a general OMAS:

- ① A suitable definition of (sequences of) points that play the role of equilibria and define a notion of stability that is suitable for them;
- ② An extension of the notion of distance between two vectors for vectors of unequal length;
- ③ Sufficient conditions for "stability".

The concept of trajectory of points of interest replaces the classical notion of equilibrium in open multi-agent systems.

Definition (Trajectory of Points of Interest (TPI))

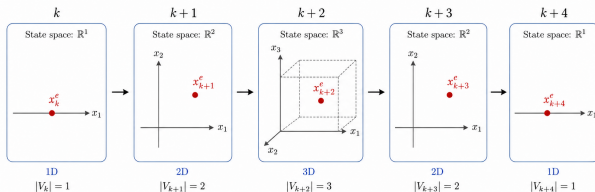
Consider an open multi-agent system. Assume that for every $k \geq 0$, the equation

$$y = \bar{f}(y, V_k, E_k, u_k)$$

has a unique solution y and denote that solution as x_k^e (the fixed point of the map $\bar{f}$). Then, the open sequence $\{x_k^e : k \in \mathbb{Z}_{\geq 0}\}$ is called trajectory of points of interest of the open multi-agent system.

Geometric intuition of a TPI

Trajectory of Points of Interest (TPI)



A TPI is an open sequence of fixed points: at each time k , x_k^e is the unique fixed point of the map associated with the current active set V_k .

The dimension of x_k^e may vary over time.

A trajectory of points of interest evolving through state spaces of different dimensions.

Key idea

A TPI is the sequence

$$\{x_k^e\}_{k \geq 0}, \quad x_k^e \in \mathbb{R}^{V_k},$$

where each x_k^e is the unique fixed point associated with the active network at time k .

- The ambient space changes when agents join or leave.
- Hence, x_k^e does not live in a fixed Euclidean space.
- The TPI replaces the classical notion of equilibrium for OMAS.

Banach Fixed Point Theorem

Let $T : X \rightarrow X$ be a nonlinear map on a complete metric space (X, d) , d is a distance function.

Contraction

The map T is a contraction if there exists a constant $c \in [0, 1)$ such that

$$d(T(x), T(y)) \leq c d(x, y), \quad \forall x, y \in X.$$

Banach Fixed Point Theorem

If T is a contraction, then:

- there exists a unique fixed point $x^* \in X$ such that $x^* = T(x^*)$;
- for any initial condition $x(0) \in X$, the iteration $x(k+1) = T(x(k))$ converges to x^* ;
- the convergence is at least geometric: $d(x(k), x^*) \leq c^k d(x(0), x^*)$.

If a discrete-time nonlinear system $x(k+1) = T(x(k))$ is contractive, then all trajectories converge to the same unique equilibrium x^* , independently of the initial condition.

The existence of a TPI is guaranteed for some classes of OMAS.

Definition (Contractive OMAS)

The OMAS is said to be contractive if there exists $\gamma \in [0, 1)$ such that for all $x, y \in \mathbb{R}^{V_k}$ and for all $k \geq 0$

$$\|\bar{f}(x, V_k, E_k, u_k) - \bar{f}(y, V_k, E_k, u_k)\| \leq \gamma \|x - y\|. \quad (7)$$

By the Banach Fixed Point Theorem, every contractive OMAS has a TPI.

The next definition introduces a notion akin to a weak form of Lyapunov stability for open multi-agent systems.

Definition (Open Stability of a Trajectory of Points of Interest)

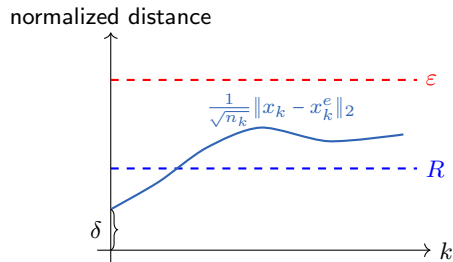
Let x_k be the state trajectory of an open multi-agent system. A trajectory of points of interest x_k^e is said to be open stable if there exists a stability radius $R \geq 0$ with the following property: for every $\varepsilon > R$, there exists $\delta > 0$ such that if $\frac{1}{\sqrt{n_0}} \|x_0 - x_0^e\|_2 < \delta$, then $\frac{1}{\sqrt{n_k}} \|x_k - x_k^e\|_2 < \varepsilon$ for every $k \geq 0$.

Euclidean distances are normalized by the number of agents.

This normalization, trivial when the set of agents is fixed, is crucial because it allows for a fair comparison of distances evaluated in spaces of different dimensions.

It makes the stability radius independent of the number of agents.

Open stability: geometric interpretation



Open stability

A TPI x_k^e is open stable if there exists $R \geq 0$ such that, for every $\varepsilon > R$, there exists $\delta > 0$ satisfying

$$\frac{1}{\sqrt{n_0}} \|x_0 - x_0^e\|_2 < \delta$$

implies

$$\frac{1}{\sqrt{n_k}} \|x_k - x_k^e\|_2 < \varepsilon, \quad \forall k \geq 0.$$

Interpretation

The normalized trajectory remains inside any tube of radius $\varepsilon > R$ around the moving TPI. The radius R is the unavoidable residual error caused by openness.

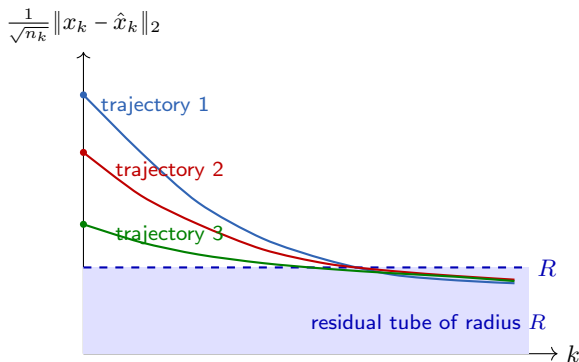
Global asymptotic stability of the TPI

Definition (Global asymptotic open stability of the TPI)

Consider an OMAS whose TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ is open stable with stability radius $R \geq 0$. The TPI is said to be “globally asymptotically open stable” w.r.t. $\|\cdot\|_2$ if all trajectories converge to within a distance of R from the TPI:

$$\lim_{k \rightarrow \infty} \frac{1}{\sqrt{n_k}} \|x_k - \hat{x}_k\|_2 \leq R.$$

Global asymptotic open stability



Interpretation

Global asymptotic open stability means that:

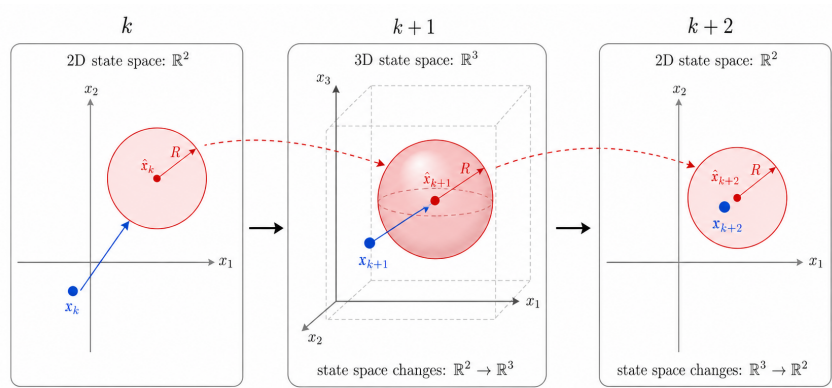
- the TPI $\{\hat{x}_k\}$ is already open stable;
- for any initial condition, the normalized distance

$$\frac{1}{\sqrt{n_k}} \|x_k - \hat{x}_k\|_2$$

eventually approaches the tube of radius R ;

- openness prevents exact convergence in general, but the asymptotic error remains bounded

Global asymptotic open stability: changing state spaces



The TPI is a moving target and the state space may change dimension.
Global asymptotic open stability means the normalized distance of x_k from $\hat{x}_k$ eventually remains within radius R .

We now define a so-called “open” distance which is used to evaluate the distance between two points with labeled components that belong to Euclidean spaces of different dimensions.

Definition (Open distance)

Let V_1 and V_2 be two finite sets of node indices. Let $d: \mathbb{R}^{V_1} \times \mathbb{R}^{V_2} \rightarrow \mathbb{R}_{\geq 0}$ be defined as

$$d(x, y) = \sqrt{\sum_{i \in V_1 \cap V_2} (x^i - y^i)^2 + \sum_{i \in V_1 \setminus V_2} (x^i)^2 + \sum_{i \in V_2 \setminus V_1} (y^i)^2}$$

for any $x \in \mathbb{R}^{V_1}$ and $y \in \mathbb{R}^{V_2}$.

In the particular case in which the two points have components with the same labels, i.e., the two OMAS have the same agents, then the open distance reduces to the Euclidean distance. Variants can be given based on norms different from the 2-norm.

Proposition (Properties of open distance functions)

The open distance function $d(x, y)$ is such that for any vectors x , y , and z of possibly different dimensions:

- ① $d(x, y) \geq 0$;
- ② $d(x, y) = d(y, x)$;
- ③ If $x = y$, then $d(x, y) = 0$;
- ④ $d(x, z) \leq d(x, y) + d(y, z)$

Proof.

Properties 1), 2), and 3) being evident, we now prove property 4), i.e., the triangle inequality.

Consider sets V_x , V_y , V_z and define the union set $R = V_x \cup V_y \cup V_z$ and new vectors $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^R$ where their generic component is defined as $\bar{x}^i = x^i$ if $i \in V_x$ and $\bar{x}^i = 0$ otherwise.

Since $\bar{x}, \bar{y}, \bar{z}$ belong to the same space $\mathbb{R}^R$, it follows that the triangle inequality

$$d(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{z}) + d(\bar{z}, \bar{y})$$

holds true since the open distance reduces to the ordinary Euclidean one.

The result follows because one can readily verify that $d(\bar{x}, \bar{y}) = d(x, y)$. □

Definition (Open sequence of bounded variation)

An open sequence $\{y_k\}$ of points $y_k \in \mathbb{R}^{V_k}$ is said to have bounded variation if there exists a constant $B \geq 0$ such that $d(y_{k+1}, y_k) \leq \sqrt{|V_{k+1}|}B$ for all $k \in \mathbb{Z}_{\geq 0}$.

This definition normalizes the open distance by the number of components of the vectors.

Thus, a TPI $\{x_k^e\}$ has bounded variation if there exists B such that $d(x_{k+1}^e, x_k^e) \leq \sqrt{|V_{k+1}|}B$ for all $k \in \mathbb{Z}_{\geq 0}$.

In order to provide a sufficient condition to ensure stability in the above sense, we will need to combine assumptions on both the associated TPI and on its arrival process, that is, on the process by which agents join the OMAS during time.

Definition (Bounded arrival process)

An arrival process is said to be bounded if there exists $H \geq 0$ such that each agent joins the OMAS with a state value such that

$$\sqrt{\sum_{i \in A_k} (x_{k+1}^i - x_{k+1}^{e,i})^2} \leq \sqrt{|V_{k+1}|} H \quad \forall k \in \mathbb{Z}_{\geq 0}$$

where $x_k^{e,i}$ denotes the v -th component of x_k^e .

We are now ready to state our main stability result.

Theorem (Stability of Open Multi-Agent Systems)

Consider an open multi-agent system with state trajectory $\{x_k\}$. Assume that

- ① *the OMAS is contractive with parameter $\gamma \in [0, 1)$;*
- ② *its TPI $\{x_k^e\}$ has bounded variation with constant B ;*
- ③ *the arrival process is bounded with constant H ;*
- ④ *$|V_{k+1}| \geq \beta^2 |V_k|$ for all k with $\beta > \gamma$.*

Then, the trajectory of points of interest is open stable with stability radius

$$R = \frac{B + H}{1 - \frac{\gamma}{\beta}}$$

M. Franceschelli, P. Frasca "Stability of Open Multi-Agent Systems and Applications to dynamic Consensus", IEEE Transactions on Automatic Control, 2021.

Proof.

At each k the agents first update their state, then some new agents may join and some may leave. By considering the open distance function, it holds

$$\begin{aligned}
 d(x_{k+1}, x_{k+1}^e) &= \sqrt{\sum_{i \in V_{k+1} \cap V_k} (x_{k+1}^i - x_{k+1}^{e,i})^2 + \sum_{i \in V_{k+1} \setminus V_k} (x_{k+1}^i - x_{k+1}^{e,i})^2} \\
 &\leq \sqrt{\sum_{i \in V_{k+1} \cap V_k} (x_{k+1}^i - x_{k+1}^{e,i})^2} + \sqrt{\sum_{i \in V_{k+1} \setminus V_k} (x_{k+1}^i - x_{k+1}^{e,i})^2} \\
 &\leq \sqrt{\sum_{i \in V_{k+1} \cap V_k} (x_{k+1}^i - x_k^{e,i})^2} + \sqrt{\sum_{i \in V_{k+1} \cap V_k} (x_{k+1}^{e,i} - x_k^{e,i})^2} + \sqrt{\sum_{i \in V_{k+1} \setminus V_k} (x_{k+1}^i - x_{k+1}^{e,i})^2}.
 \end{aligned} \tag{8}$$

□

Proof.

Since the OMAS is contractive, we observe that

$$\sqrt{\sum_{i \in V_{k+1} \cap V_k} (x_{k+1}^i - x_k^{e,i})^2} \leq \gamma \sqrt{\sum_{i \in V_{k+1} \cap V_k} (x_k^i - x_k^{e,i})^2} \leq \gamma d(x_k, x_k^e). \quad (9)$$

Now, note that the trajectory of points of interest is of bounded variation, implying

$$\sqrt{\sum_{i \in V_{k+1} \cap V_k} (x_{k+1}^{e,i} - x_k^{e,i})^2} \leq d(x_{k+1}^e, x_k^e) \leq \sqrt{|V_{k+1}|} B, \quad (10)$$

and that the arrival process is bounded as per Definition 9, implying

$$\sqrt{\sum_{i \in V_{k+1} \setminus V_k} (x_{k+1}^i - x_{k+1}^{e,i})^2} \leq \sqrt{|V_{k+1}|} H. \quad (11)$$

Thus, by upper bounding the righthand side of (8) by (9)-(10)-(11), we can write

$$d(x_{k+1}, x_{k+1}^e) \leq \gamma d(x_k, x_k^e) + \sqrt{|V_{k+1}|} B + \sqrt{|V_{k+1}|} H. \quad (12)$$



Proof.

Let us now divide both sides of (12) by the square root of the cardinality of $|V_{k+1}|$

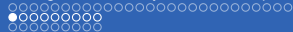
$$\frac{d(x_{k+1}, x_{k+1}^e)}{\sqrt{|V_{k+1}|}} \leq \gamma \frac{d(x_k, x_k^e)}{\sqrt{|V_{k+1}|}} + B + H.$$

By assumption, $|V_{k+1}| \geq \beta^2 |V_k|$ where $\beta > \gamma$, thus we can write

$$\frac{d(x_{k+1}, x_{k+1}^e)}{\sqrt{|V_{k+1}|}} \leq \frac{\gamma}{\beta} \frac{d(x_k, x_k^e)}{\sqrt{|V_k|}} + B + H.$$

By this inequality, it follows that the TPI is open stable with stability radius $R = \frac{B+H}{1-\frac{\gamma}{\beta}}$.





Convergence requires that the arrival/departure process guarantees some good behavior of the sequence of graphs.

For every $k \in \mathbb{Z}_{\geq 0}$:

- 1 Graph $G(k)$ is undirected;
- 2 $\max_k d_{max}(k) \leq \frac{1}{2\varepsilon}$ for all $i \in V_k$;
- 3 $\beta^2 \leq \frac{|V_{k+1}|}{|V_k|}$ for some positive scalar β .

Theorem (Stability of Open Proportional Dynamic Consensus)

Consider the Open Proportional Dynamic Consensus algorithm (OPDC) under Assumptions (slide 57) and assume that $\beta > 1 - \alpha$.

Let $\min_k \lambda_2(L_k) \geq \underline{\lambda} \geq 0$ and $\bar{u}_k = \frac{\mathbf{1}^\top u_k}{|V_k|} \mathbf{1}$, $\hat{u}_k = u_k - \bar{u}_k$. If the sequence of inputs $\{u_k\}$ satisfies

$$\|\hat{u}_k\|_\infty \leq \Pi, \quad \Pi \geq 0 \quad (13)$$

and

$$d(\bar{u}_{k+1}, \bar{u}_k) \leq \sqrt{|V_{k+1}|} U, \quad U \geq 0, \quad (14)$$

then the OPDC is open stable with stability radius

$$R = \frac{B + H}{1 - \frac{\gamma}{\beta}} = \frac{\left(1 + \frac{2}{1 + \frac{\varepsilon}{\alpha} \underline{\lambda}} + \frac{1}{\beta} \frac{1}{1 + \frac{\varepsilon}{\alpha} \underline{\lambda}}\right) \Pi + U}{1 - \frac{1 - \alpha}{\beta}}$$

The proof is divided into four steps which lead to the application of the general Theorem which establishes existence of a stability radius.

The Laplacian matrix of an undirected graph is symmetric and positive semi-definite, and its eigenvalues lie in the interval $(0, 2d_{max}(k))$.

Since $\rho(k) < 1$ at each k the OMAS is contractive with respect to the Euclidean norm with factor $\rho(k) = \gamma = |1 - \alpha|$, independent from k .

Since the OMAS is contractive, the TPI is unique according to the Banach fixed-point Theorem.



Proof.

Since the eigenvector corresponding to the largest eigenvalue of $(\alpha I + \varepsilon L_k)^{-1}$ is $\mathbf{1}$ and $\mathbf{1}^T \hat{u}_k = 0$:

$$\|x_k^e - \bar{u}_k\|_2 = \|\alpha(\alpha I + \varepsilon L_k)^{-1} \hat{u}_k\|_2 \leq \frac{1}{1 + \frac{\varepsilon \lambda_k}{\alpha}} \|\hat{u}_k\|_2, \quad (16)$$

where $(1 + \frac{\varepsilon \lambda_k}{\alpha})^{-1}$ is the second largest eigenvalue of $\alpha(\alpha I + \varepsilon L_k)^{-1}$. Then, the distance between the point of interest and the reference signal at time k satisfies

$$d(x_k^e, \bar{u}_k) = \|x_k^e - \bar{u}_k\|_2 \leq \frac{\alpha}{\alpha + \varepsilon \lambda_k} \|\hat{u}_k\|_2 \leq \frac{1}{1 + \frac{\varepsilon}{\alpha} \lambda_k} \sqrt{|V_k|} \|\hat{u}_k\|_\infty.$$

Since $\frac{|V_{k+1}|}{\beta^2} \geq |V_k|$, $\|\hat{u}_k\|_\infty \leq \Pi$ and $d(\bar{u}_{k+1}, \bar{u}_k) \leq \sqrt{|V_{k+1}|} U$ for $U \geq 0$, from the properties of the open distance function:

$$d(x_{k+1}^e, x_k^e) \leq \sqrt{|V_{k+1}|} \left(\frac{1}{1 + \frac{\varepsilon}{\alpha} \lambda} \left(1 + \frac{1}{\beta} \right) \Pi + U \right) = \sqrt{|V_{k+1}|} B.$$

$$B = \frac{1}{1 + \frac{\varepsilon}{\alpha} \lambda} \left(1 + \frac{1}{\beta} \right) \Pi + U.$$

Step 3: Prove that the arrival process of the OPDC is bounded with constant H .

and $|u_{k+1}^v - \bar{u}_{k+1}| \leq \Pi$ by assumption, by recalling that $A_{k+1} = V_{k+1} \setminus V_k$, it holds

$$\sqrt{\sum_{i \in A_{k+1}} (x_{k+1}^i - x_{k+1}^{e,i})^2} \leq \sqrt{|V_{k+1} \setminus V_k|} (1 + \frac{\alpha}{\alpha + \varepsilon \lambda_{k+1}}) \Pi \leq \sqrt{|V_{k+1}|} (1 + \frac{1}{1 + \frac{\varepsilon \lambda}{\alpha}}) \Pi$$

Thus, the arrival process of the OPDC algorithm is bounded according to with $H = \left(1 + \frac{1}{1 + \frac{\varepsilon\lambda}{\alpha}}\right)\Pi$. $\square$

Proof.

Step 4: Apply the Theorem (Stability of Open MAS). It follows that the TPI of the OPDC algorithm is open stable with stability radius $R = \frac{B+H}{1-\frac{\gamma}{\beta}}$.



$$R_0 = \frac{(3\beta + 1)\Pi + \beta U}{\alpha + \beta - 1}.$$

Both D_{max} and D_{min} make evident that if the inputs have large variations (II is large), then the stability

This is a pre-proof of the article accepted for publication in *Journal of Management Education*. The final version of the article is available at: <https://doi.org/10.1177/0095647219854166>

Consequences of the demands of the working Assumptions:

- 1 *Undirected networks.* Our general Open Multi-Agent framework is defined for the more general case of directed graphs. However, the application to Dynamic Consensus is done for undirected graphs, because this assumption guarantees the OPDC to be a contractive OMAS.
- 2 *Bounded degrees.* Limiting the degrees of the agents may be detrimental to the connectivity for proximity graphs and therefore increase the stability radius. No problem for random k-regular graphs, line, ring, grid graphs and others.
- 3 *Bounded rate of departure.* Due to this assumption, if the rate of departure is large, then β cannot be chosen too large, which makes the radius larger.

All these conditions are stated for all times: this uniformity makes the result rather conservative, as will be apparent in the next simulations.

- Minimum algebraic connectivity $\underline{\lambda} = 0.9037$,
- $|V_{k+1}| \geq \beta^2 |V_k|$ with $\beta = 0.9975$,
- Maximum degree: $\max_k d_{max} = 20$,
- Maximum variations of the inputs to the MAS: $\Pi = 0.5139$, and $U = 0.0001785$.
- Bound on TPI: $B = 0.7900$,
- Arrival process: $H = 0.9083$,
- Stability radius $R = 17.375$, (conservative estimate).

Concluding remarks

- The paradigm of **Open Multi-Agent Systems** applies to all realistic scenarios where agents join and leave network systems seamlessly.
- While real networks have always been open, the field of control theory usually neglects the transient behavior during departure and arrival events: there is a **lack of formal guarantees**.
- We have discussed first attempts at making distributed algorithms work in open networks by accounting for arrivals and departures explicitly and **preserving invariant quantities during changes of the network** size.
- We have discussed how to formalize the "openness" in the context of dynamical systems and what questions we can ask to Open MAS, i.e., how to tailor bounds on the behavior of the network to ensure that the actual behavior is close to the desired ones, explicitly accounting for arrival and departures.
- There are many **low-hanging research opportunities** that can be picked simply by asking: **"What happens to this method when the network changes size?"**

Future directions in Open Multi-Agent Systems I

- **Persistent openness:** notions of practical stability without eventual network closure.
- **Directed networks:** unbalanced graphs, weak connectivity, and local knowledge only.
- **Asynchronous protocols:** local clocks, event-triggered updates, delays, and packet losses.
- **Unannounced departures:** detection, compensation, watchdog mechanisms, and false alarms.
- **Nonlinear OMAS:** nonlinear couplings, nonlinear agent dynamics, networks of coupled oscillators, and hybrid systems.
- **Heterogeneous agents:** different dynamics, sensing capabilities, computation, and communication constraints.
- **Performance limits:** lower bounds on error, convergence rate, memory, and communication cost.

Future directions in Open Multi-Agent Systems II

- **Open optimization and learning:** time-varying data, objectives, constraints, and agents.
- **Scalable implementations:** low-memory protocols, randomized interactions, and decentralized initialization.
- **Game theory:** strategic arrivals/departures, incentives, Nash equilibria in open games, and learning in open games.
- **Security and resilience:** adversarial agents, denial-of-service attacks, spoofing, and corrupted data.
- **Privacy-preserving OMAS:** anonymous networks, private values, and privacy-aware consensus.
- **Energy systems:** smart grids, demand response, EV charging, microgrids, and energy communities.
- **Robotics and mobility:** swarms, vehicle networks, connectivity maintenance, and collision avoidance.



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Introduction to Open Multi-Agent Systems - Part II

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